Transcendental Signature Sequences

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Signature Sequences

For a positive irrational number x, its signature sequence is created by forming $y = p + qx$ for positive integers p and q (Kimberling, 1997). Then, the y's are sorted in increasing order and the corresponding set of p's is the signature sequence of x. This construction follows from the definition of an irrational number: if x is irrational, then it cannot be written as a ratio of integers, so $x \neq p/q$ (where p and q are integers). Rearranging, $0 \neq qx - p$; without loss of generality, $y = p + qx$, and y will never be 0. Moreover, for two pairs of integers (p_1, q_1) and (p_2, q_2), y_1 will only equal y_2 if and only if $p_1 = p_2$ and $q_1 = q_2$. Consequently, every different pair of integers (p, q) will yield a different value of y. These y values can be sorted and the resulting sequence of p's is the signature sequence.

The notion of a signature sequence can be extended to transcendental numbers. A transcendental number is one that is not algebraic, an algebraic number being one that is a solution to a polynomial equation with integer coefficients. For example, the polynomial equation $x^3 - x^2 - 8x + 8$ has solutions $x = 1$, $-\sqrt{8}$, and $\sqrt{8}$, all of which are algebraic numbers. A transcendental number would not be a solution to any such equation (with integer coefficients), no matter what degree. Examples of transcendental numbers are e (approximately 2.71828) and π (approximately 3.14159).

Following the logic of the signature sequence, similar sequences for higher-order polynomials can be created for positive transcendental numbers x. Since x is transcendental, $y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \neq 0$ for integer coefficients $a_0, a_1, a_2, \ldots a_n$. The y's can be sorted and the sequence of a_0's is the desired sequence. (Note, this applies for $x > 1$. If $x < 1$, then the process is the same, but the a_n coefficients form the sequence of interest, as discussed below). In this work, $S_n(x)$ is the signature-type sequence for transcendental x, using integer polynomials of degree n. $S_1(x)$ is the standard signature sequence. The limit of $S_n(x)$ as n approaches infinity is termed $T(x)$, the transcendental signature sequence. The question of this work is, what are the characteristics of $T(x)$?

$x = e$, the natural base (approximately 2.71828)

The signature sequence for e is sequence A023123 in the Online Encyclopedia of Integer Sequences (Sloane) and begins: 1, 2, 3, 1, 4, 2, 5, 3, 6, 1, 4, 7, 2, 5, 8, 3, 6, 9, 1, 4. In the present terminology, this sequence is $S_1(e)$. $S_2(e)$ begins the same way and deviates after the 16th term. S_3 begins the same as S_2, but deviates from S_2 after 136 terms. S_4 starts out like S_3, up through 2,389 terms. And S_5 mirrors S_4 for the first 89,458 terms. Clearly, the number of terms that remain the same increases greatly with the degree of the polynomial used. For e, the first several hundred thousand terms of $T(e)$ can be obtained from S_5.

For any x, sequences for higher degrees have early terms that are identical to those of $T(x)$. This can be seen through the construction process. In generating $S_{n+1}(x)$, the initial y values will be the same as those for $S_n(x)$, except that they will be larger by x^{n+1} (corresponding to a coefficient of $a_{n+1} = 1$ for the x^{n+1} term). This increment won't change the sort order of the terms until $2x^{n+1}$ becomes relatively large. If $x > 1$, then larger n will make x^{n+1} much larger, delaying the onset of the deviation of $S_{n+1}(x)$ from $S_n(x)$.

Figures 1 and 2 show the graphs of the first 1000 terms of the standard signature sequence of e and of its transcendental signature sequence, respectively.

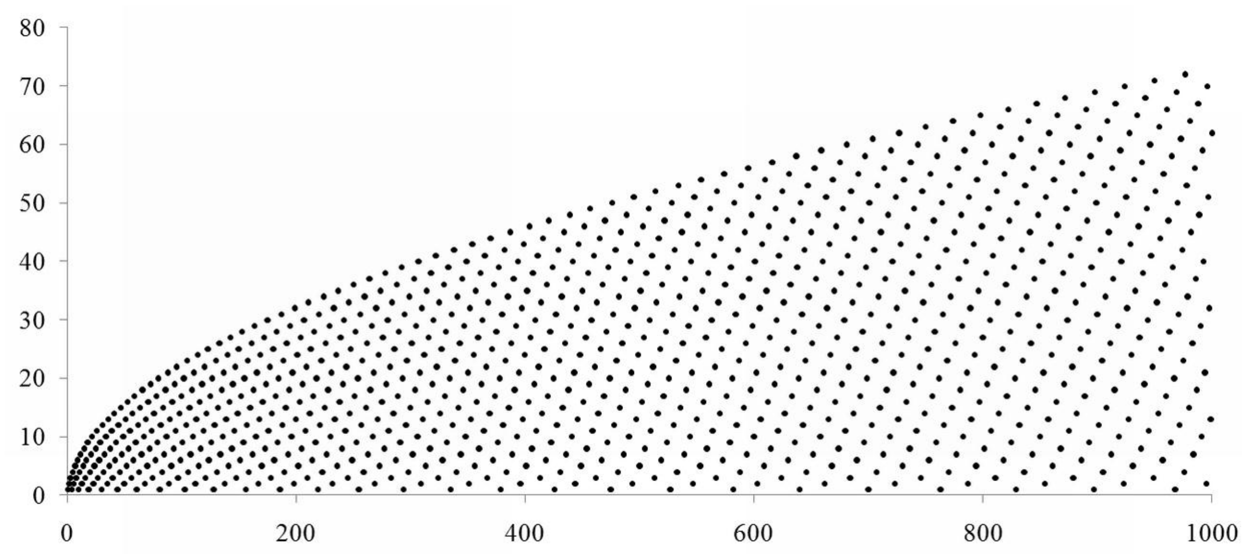

Figure 1: Signature sequence of e, $S_1(e)$.

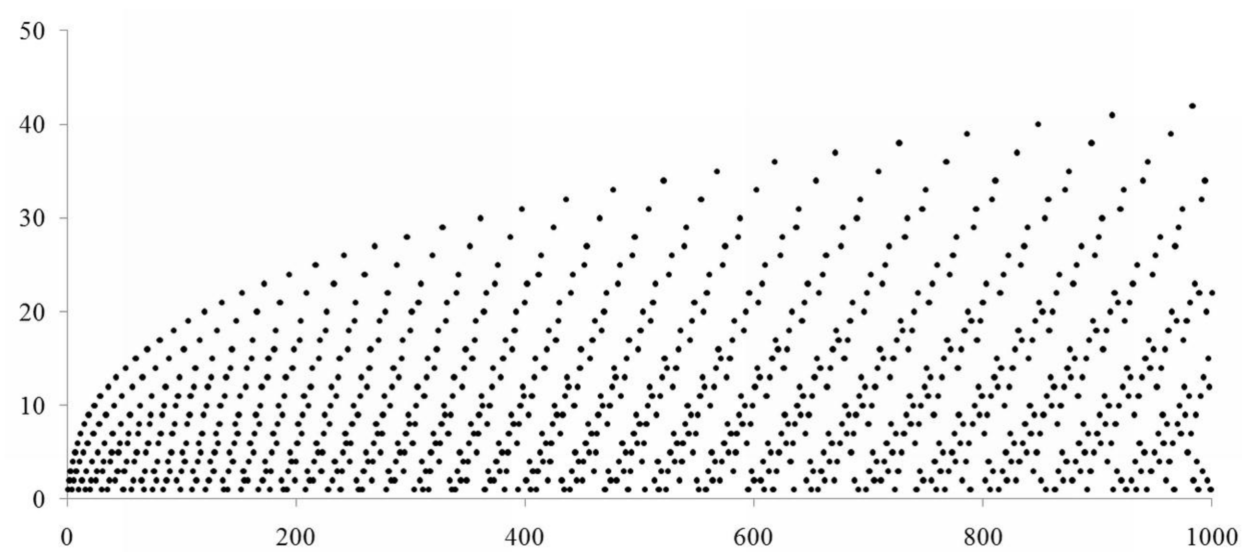

Figure 2: Transcendental signature sequence of e, $T(e)$.

While the two graphs are superficially similar, the S_1 sequence is very regular in terms of the spacing of the points. T appears to be organized in bands, from somewhat regular near the top to quite irregular at the bottom. Figures 3 and 4 show detail of the sequences from indices 10,000 through 11,000, highlighting the differences in structure.

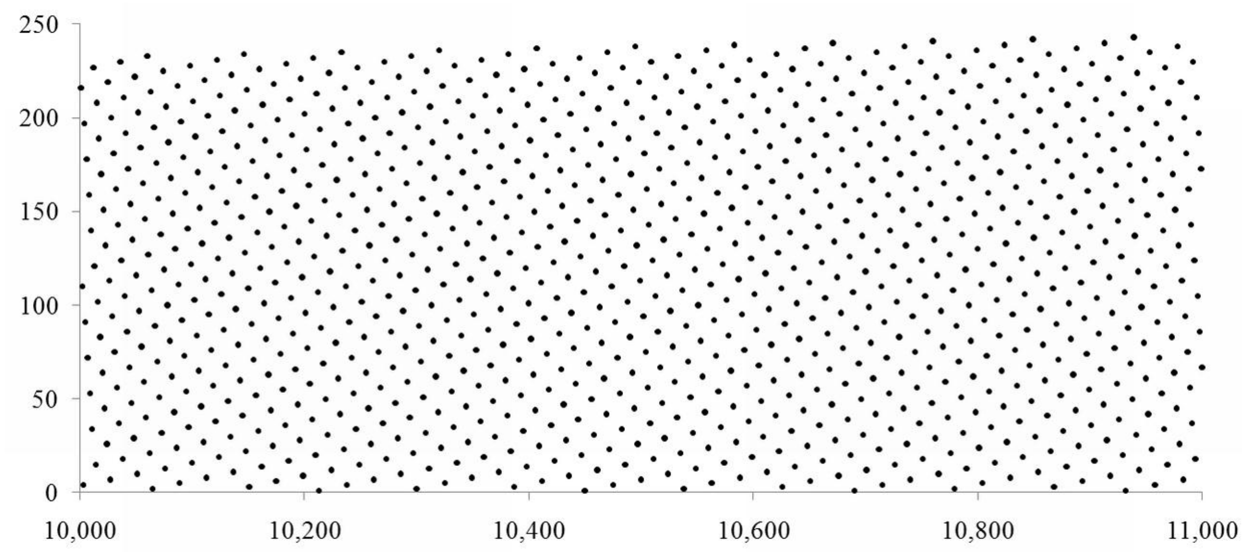

Figure 3: Detail of $S_1(e)$.

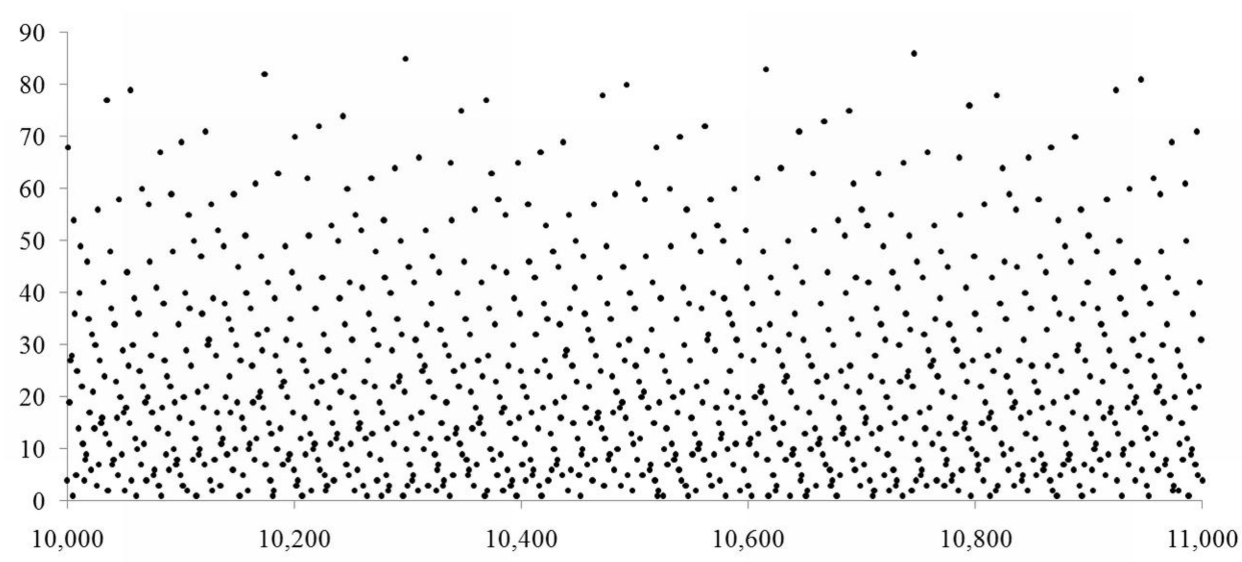

Figure 4: Detail of $T(e)$.

These graphs suggest that T is qualitatively different from S_1, and this is confirmed by looking at T's associative array. Kimberling states that in order for a sequence to be fractal, its associative array must be an interspersion. That is, following the path of increasing index numbers in the body of the array, if a lower row is left to visit an upper row, then every row above it must be visited once and only once before returning to the original row. Table 1 shows the upper-left corner of the associative array for $T(e)$. In row 3, the 26^{th} and 31^{st} elements of the sequence are both 3 (bolded and italicized in the table). Between those two elements, the sequence does not take the value of 2. Further, T is 7 at the 12^{th} and 23^{rd} places (row 7), between which it is 2 twice (elements 13 and 21 in row 2).

Table 1: Upper-left corner of the associative array for $T(e)$, showing how $T(e)$ is not a fractal sequence.

	Count									
Value	1	2	3	4	5	6	7	8	9	10
1	1	4	10	16	20	30	35	49	56	67
2	2	6	***13***	***21***	25	37	42	58	65	78
3	3	8	17	***26***	***31***	44	50	68	76	90
4	5	11	22	32	38	52	59	79	88	103
5	7	14	27	39	45	61	69	91	101	117
6	9	18	33	46	53	71	80	104	115	132
7	***12***	***23***	40	54	62	82	92	118	130	149
8	15	28	47	63	72	94	105	133	147	168
9	19	34	55	73	83	107	119	150	165	188
10	24	41	64	85	95	122	134	169	185	211

While not a fractal sequence in the strict sense of Kimberling, a fractal structure is suggested by the graphs and by the ways in which one order $S_n(e)$ leads to the next, $S_{n+1}(e)$.

Progression from $S_n(e)$ to $S_{n+1}(e)$

As discussed above, the initial terms of $S_n(x)$ are the same as the initial terms of $T(x)$. However, when comparing $S_n(x)$ to $S_{n+1}(x)$, it seems that the latter can be built by inserting terms of itself into the former. For example, Table 2 describes the first 41 terms of $S_1(e)$ and corresponding terms from $S_2(e)$. The first 10 terms of each are listed explicitly and both sequences are the same for the first 15 terms. The next term of S_2 is different, but then the following four terms of S_2 match terms 16 – 19 of S_1. Continuing, the two sequences have runs of equal terms interspersed with new terms inserted into S_2. The first 10 of the new terms are: 1, 2, 3, 1, 4, 2, 5, 3, 6, and 1, the same as the first 10 terms of both S_1 and S_2.

Table 2: Differences between the initial terms of $S_1(e)$ and $S_2(e)$.

S_1 Index	S_1	S_2
1	1	1
2	2	2
3	3	3

4	1	1
5	4	4
6	2	2
7	5	5
8	3	3
9	6	6
10	1	1
11 – 15	Same	
--	--	1
16 – 19	Same	
--	--	2
20 – 23	Same	
--	--	3
24 – 26	Same	
--	--	1
27	Same	
--	--	4
28 – 31	Same	
--	--	2
32	Same	
--	--	5
33 – 36	Same	
--	--	3
37	Same	
--	--	6
38 – 39	Same	
--	--	1
40 – 41	Same	

Empirical results suggest that the deviation from one generation to the next drops off quickly with increasing degree of the underlying polynomials. Comparing $S_1(e)$ with $S_2(e)$, the first 45,421 terms of S_1 corresponded to the first 1,041,032 terms of $S_2(e)$, with 995,611 terms inserted into $S_2(e)$. These matched exactly the first 995,611 terms of $S_2(e)$. Table 3 shows the numbers of terms that were the same and of inserted terms when $S_n(e)$ was compared with $S_{n+1}(e)$ over the first approximately 1 million terms.

Table 3: Differences between $S_n(e)$ and $S_{n+1}(e)$

From	To	Total Terms	Same Terms	Inserted Terms	First Insertion
S_1	S_2	1,041,032	45,421	995,611	16
S_2	S_3	1,022,194	236,677	785,515	137
S_3	S_4	1,015,333	638,355	376,978	2,390
S_4	S_5	1,040,000	1,005,918	34,082	89,459

As the degree increases, moving $S_n(e)$ closer to $T(e)$, the similarity between $S_n(e)$ and $S_{n+1}(e)$ increases. The proportion of consistent terms and the point of insertion of the first term both increase exponentially with the degree squared.

Transcendental Signature Sequences for other numbers

Generally speaking, the features observed for *e* carry over to the transcendental signature sequences (and the $S_n(x)$ sequences leading up to them) for other values of *x*. The convergence seems to speed up with larger *x*, which stands to reason, as larger *x* leads to larger x^n (for positive n) and less impact on the sort order of initial *y* values. Figure 5 shows the growth in the numbers of consistent terms when moving from $S_n(x)$ to $S_{n+1}(x)$ for four different transcendental *x* values: ln(3)/ln(2) (approximately 1.585), e, π, and Feigenbaum's delta (approximately 4.6692).

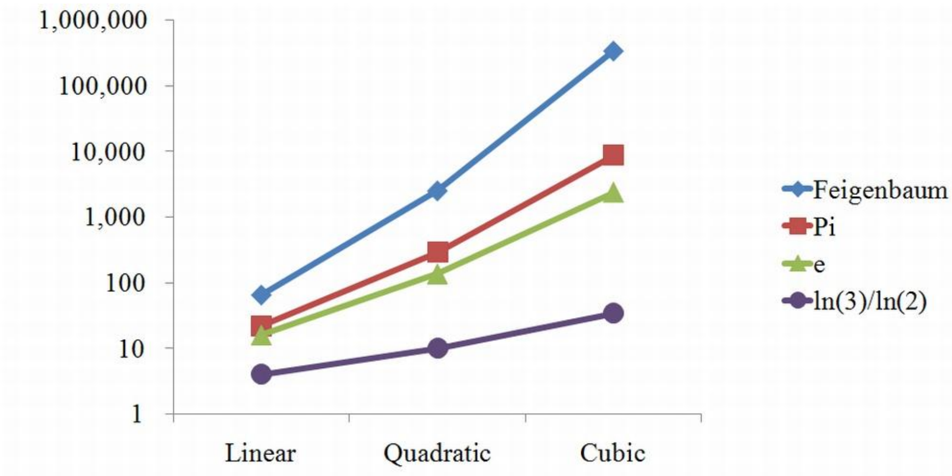

Figure 5: Numbers of consistent terms between $S_n(x)$ and $S_{n+1}(x)$ for four different transcendental values of *x*.

For positive transcendental numbers less than 1, the same process can be undertaken, but the sequence of the leading coefficients is the sequence of interest, not the constant terms. Recalling that:

$y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$,

x^n can be factored out of each term, leaving:

$y = (x^n)(a_0x^{-n} + a_1x^{1-n} + a_2x^{2-n} + \ldots + a_n)$, or
$y = (x^n)(a_0z^n + a_1z^{n-1} + a_2z^{n-2} + \ldots + a_n)$, where z = 1/*x*. Rewriting,
$y = (x^n)(a_n + a_{n-1}z + a_{n-2}z^2 + \ldots + a_0z^n)$.

Since *x* is positive, multiplying each *y* by x^n doesn't change the sort order. Thus, the *y*'s can be sorted and the a_n coefficients (instead of a_0) be taken as $S_n(z) = S_n(1/x)$.

Making this change, the sequences exhibit the same basic patterns as do the constant-term $S_n(x)$

for $x > 1$. The smaller x (or larger z), the faster the sequences converge to $T(x)$.

References

Kimberling, C. “Fractal Sequences and Interspersions.” Ars Combin. 45, 157-168, 1997.

Weisstein, Eric W. “Transcendental Number.” From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/TranscendentalNumber.html.