Spirolateral-Type Images from Integer Sequences

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Abstract

The ideas underlying spirolaterals can be extended using sequences of positive integers, resulting in several new classes of images. These images can be open or closed, symmetric or asymmetric, simple or complex. Examples are shown using digit sum, Kolakoski, and Fibonacci word sequences.

Spirolaterals

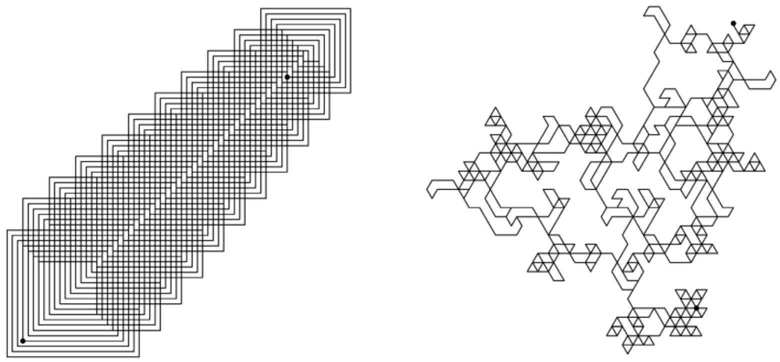
Spirolaterals are constructed by drawing a series of segments from an origin, turning by the same amount after each segment. The lengths of the segments increase through the sequence 1, 2, 3, ..., n, and then start again at 1. Typically, the angle turned is the same after each segment, but reversals, where the turn is in the opposite direction, are allowed. Spirolaterals are generally shown as closed curves, but they need not close. See Weisstein [1] and Krawczyk [2] for examples.

The spirolateral creation process can be abstracted by allowing the segment lengths and the turn angles to vary in more general ways. In this work, either or both are determined based on infinite sequences of positive integers.

Digit Sum Sequences

The digit sum of a non-negative integer is simply the sum of its digits. For example, the digit sum of 35 is 3 + 5 = 8. Clearly, the digit sum depends on the base in which the number is expressed. For base 10, the sequence of digit sums begins: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 1. Additional terms can be found in the Online Encyclopedia of Integer Sequences (OEIS) [3]. For integers in binary, the digit sum is also the count of 1's in the representation. This sequence begins: 0, 1, 1, 2, 1, 2, 2, 3, 1, and is sequence A000120 in the OEIS [4].

To create a spirolateral-type image from a digit sum sequence, the terms can be used as the lengths of the segments. Alternatively, the turn angle can be some pre-determined base angle (say, 45°), multiplied by the sequence terms. Or, both techniques can be used simultaneously. Figure 1 shows two examples. In the left panel, the decimal digit sum is the segment length and the turns are all 90°. The curve begins at the black dot in the upper right corner and ends at the black dot in the lower left corner. In the right panel, the segments are unit length and the turn angles are multiples of 60°, with the binary digit sum being the multiplier. This curve begins in the lower right corner dot and ends in the upper right corner at the dot.



a) Base 10, segments are sequence terms, turns are all 90°

b) Base 2, segments 1 unit long, turns are 60° × sequence term

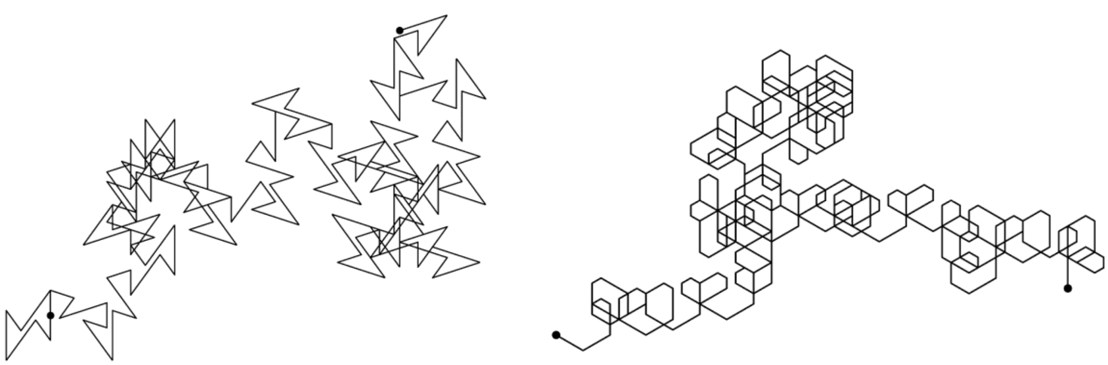
Figure 1: *Examples using Digit Sum Sequences.*

Kolakoski Sequences

The Kolakoski sequence is an infinite, self-describing sequence using only the characters 1 and 2. It begins: 1, 2, 2, 1, 1, 2, 1. Each numeral represents both a symbol and a count of how many symbols are in that run of single or double 1s and 2s. For example, the first 1 means that the first run is one character long, namely, the first 1. Next, the second character is 2. This means that the second run is two characters long, namely, 2, 2. The second 2 refers to the third run, which is two 1s, and so on. Weisstein [5] and the OEIS [6] have more information on the sequence. Variations can be constructed by having an "alphabet" with more characters and beginning with different numerals. For example:

- Using 1 and 2, beginning with 2: 2, 2, 1, 1, 2, 1, …
- Using 1, 2, and 3, beginning with 1: 1, 2, 2, 3, 3, 1, 1, 1, …
- Using 1 – 4, beginning with 4: 4, 4, 4, 4, 1, 1, 1, 1, 2, 2, 2, 2, …

Any of these sequences can be employed in the same manner as the digit sum sequence. In Figure 2, the left panel shows a curve where the 2, 2, 1, 1, 2, 1,… sequence is used for the segment lengths and the turns are 108° × the 1, 2, 2, 1, 1, 2, 1, … sequence. The image in the right panel uses sequence 3, 3, 3, 1, 1, 1, 2, 2, 2, … for the segment lengths and constant 60° turns.



a) 2, 2, 1, 1, 2, 1, … for the segment lengths and 108° × 1, 2, 2, 1, 1, 2, 1, … for the turns

b) 3, 3, 3, 1, 1, 1, 2, 2, 2, … for the segment lengths and constant 60° turns

Figure 2: *Examples using Kolakoski Sequences.*

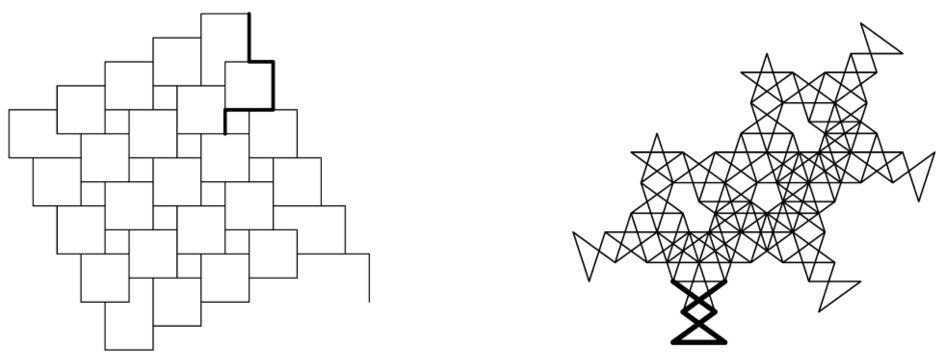
Fibonacci Word Sequences

The Fibonacci word is an infinite sequence using two symbols, like the original Kolakoski sequence. Using 1 and 2 as the two symbols, the word is built in stages:

1. $f_1 = 1$
2. $f_2 = 2$
3. $f_3 = 21$ (f_2 concatenated with f_1)
4. $f_4 = 212$ (f_3 concatenated with f_2)
5. $f_5 = 21221$ (f_4 concatenated with f_3).

After the first stage, the initial symbols of each stage are the same. When taken to its infinite limit, the constant initial substring is the Fibonacci word. Note that the length of each stage is a Fibonacci number. Monnerot-Dumaine [7] discusses the Fibonacci word and provides a method to construct fractals using it.

In the examples shown in Figure 3, a slightly different approach was taken. Instead of using the sequence terms directly, they were used as labels. In the left panel, each "1" in the sequence meant, turn left 90° and draw a segment 1 unit long. Each "2" meant, turn right 90° and draw a segment 2 units long. The first five steps (21221) are bolded and begin at the top. The right panel uses instructions of: 1: turn left 108° and draw a 1-unit segment, 2: turn left 216° and draw a 1-unit segment.



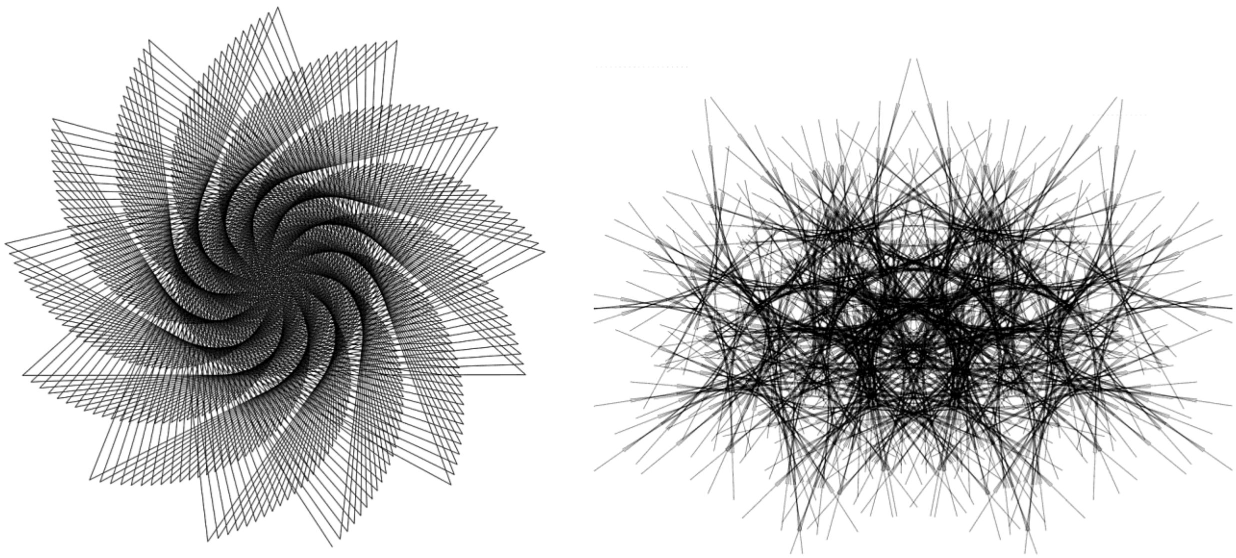
a) 1: turn left 90° and draw a 1-unit segment
2: turn right 90° and draw a 2-unit segment

b) 1: turn left 108° and draw a 1-unit segment
2: turn left 216° and draw a 1-unit segment

Figure 3: *Examples using Fibonacci Word Sequences.*

Aesthetic Considerations

For more aesthetically interesting examples of these methods, see Figure 4 and the author's "Bagpipe Jazz" [8] and "Me and My Shadow" [9]. This technique is quite powerful and can yield a fascinating variety of images, depending upon the characteristics of the sequences used. Of course, color can be added in many different ways, such as applying a spectrum along the curve, from the first segment to the last. Color can be applied according to which segment covered the pixel first or last. And in the case of busy images like in Figure 4b, colors or shading can be used to emphasize or fade segments, reducing the apparent clutter.



a) Digit sum b) Fibonacci word

Figure 4: *Additional Examples*

References

[1] E.W. Weisstein, *Spirolateral*. From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/Spirolateral.html.

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[8] L.K. Mitchell, *Bagpipe Jazz*, http://kerrymitchellart.com/gallery18/bagpipejazz.html.

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