Sequences and Patterns Arising from Mancala on an Infinite Board

Infinite Mancala

The game being investigated here is an infinite version of Mancala. In standard Mancala, each of the two players sows stones from one of their six pits, one stone in each subsequent pit. If their final stone lands in their home pit, the player takes another turn. The layout of a standard Mancala board is shown in Figure 1.

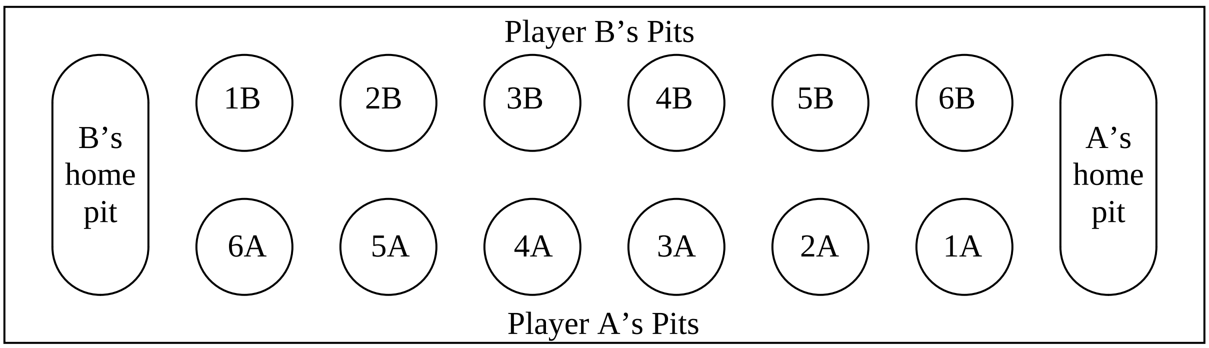

Figure 1: Layout of Standard Mancala Board

This work varies in three aspects. Firstly, the board is considered to have an unlimited number of pits (and stones). Players still choose a pit and sow stones into subsequent pits, toward the home pit. Likewise, placing the final stone into the home pit generates a free turn. The second variation is that this work only addresses the free turns. That is, what is the distribution of stones (and the sequence of harvesting pits) such that a player gets free turn after free turn, until all of her stones have been sown into her home pit? Thirdly, this investigation is limited to the maximal cases—the maximum number of stones that can be harvested using a given number of pits.

Maximum Stones

It has been shown [1] that, given an unlimited number of pits, any number of stones can be harvested. The distributions and harvesting sequences for the first few cases are shown in Table 1.

Stones	Pit 6	Pit 5	Pit 4	Pit 3	Pit 2	Pit 1	Harvesting Sequence
1*	0	0	0	0	0	1	1
2	0	0	0	0	2	0	2, 1
3*	0	0	0	0	2	1	1, 2, 1
4	0	0	0	3	0	1	1, 3, 1, 2

5*	0	0	0	3	1	1	1, 3, 1, 2, 1
6	0	0	4	2	0	0	4, 1, 3, 1, 2, 1
7	0	0	4	2	0	1	1, 4, 1, 3, 1, 2, 1
8	0	0	4	2	2	0	2, 1, 4, 1, 3, 1, 2, 1
9*	0	0	4	2	2	1	1, 2, 1, 4, 1, 3, 1, 2, 1
10	0	5	3	1	1	0	5, 1, 2, 1, 4, 1, 3, 1, 2, 1
11*	0	5	3	1	1	1	1, 5, 1, 2, 1, 4, 1, 3, 1, 2, 1
12	6	4	2	0	0	0	6, 1, 5, 1, 2, 1, 4, 1, 3, 1, 2, 1

Table 1: Initial Distribution of Stones and Harvesting Sequence for up to 12 Stones

The starred entries in the table above correspond to cases with the maximum possible stones for that number of pits. These are the case of interest. For up through 20 pits, the sequence is:

1, 3, 5, 9, 11, 17, 21, 29, 33, 41, 47, 57, 59, 77, 81, 101, 107, 117, 131, 149.

This is sequence A007952 in the Online Encyclopedia of Integer Sequences [2]. (That sequence begins with 0 instead of 1, which is in turn A002491(n) – 1. So, for n pits in the current work, the maximum number of stones that can be harvested is A002491(n + 1) – 1.)

For large numbers of pits, various trends in the distribution of stones become clear. Figure 2 shows the normalized distribution of stones for the case of 10,000 pits. Both the pit number (abscissa) and the number of stones per pit (ordinate) have been normalized by the number of pits.

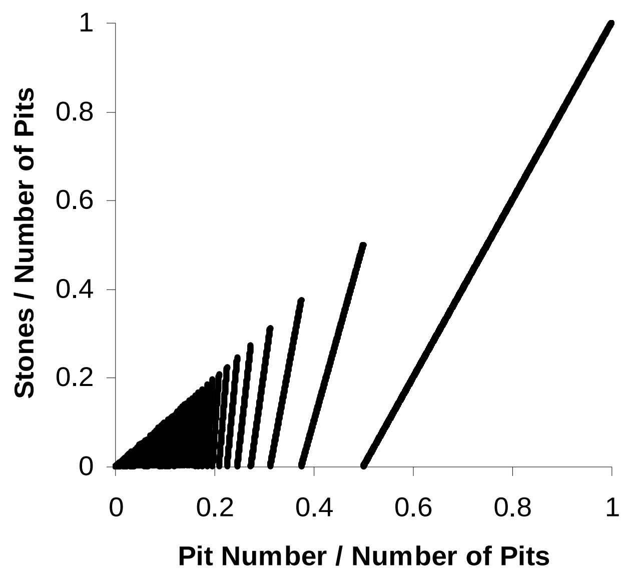

Figure 2: Distribution of Stones for 10,000 Pits

The graph can be broken into two zones. On the right, from about 0.2 on the abscissa outward, the graph consists of a series of diagonal segments with decreasing positive slope. The largest segment begins near (0.5, 0) and reaches (1, 1), with a slope of 2. This indicates that the number of stones increases by 2 for each pit in the far half of the board, ending with n stones in the last of *n* pits. The last pit must be full (a “full” pit is the n^{th} pit from the home pit and contains *n* stones) because there are no further pits to contribute to it. Consider the case of 100 pits. Pit 100 must be full with 100 stones. Pit 99 can either be harvested before or after pit 100. If it is harvested before, then when pit 100 is harvested, pit 99 will have one stone with no chance of making it to the home pit. Thus, pit 100 must be harvested before pit 99, which means that pit 99 will have one additional stone when it is harvested. Consequently, it must begin with 98 stones. This explains the increment of two stones between pits.

Moving toward the home pit, the segments increase in slope by 2. Since they are encompassed in an envelope with a 45-degree slope, the segments get shorter as well. The normalized endpoints of the last 10 segments are rational and are given in the table below.

Slope	**Endpoint**	**Numerator**	**Denominator**
2	0.500000000	1	2
4	0.375000000	3	8
6	0.312500000	5	16
8	0.273437500	35	128
10	0.246093750	63	256
12	0.225585938	231	1024
14	0.209472656	429	2048
16	0.196380615	6435	32,768
18	0.185470581	12,155	65,536
20	0.176197052	46,189	262,144

Table 2: Endpoints of Line Segments in the Normalized Stone Distribution

Moving to the left, the segments become steeper and smaller until they reach a point where non-linear behavior takes over. The location of this point increases with the number of pits: from $x = 0.0586$ for 1024 pits (pit 60) to $x = 0.0260$ for 10,000 pits (pit 260). The location of the threshold point varies approximately with the number of pits to the 0.68 power. As the number of pits increases to infinity, the pit number where the linear segments begins increases to infinity, but the relative length (x) decreases to zero. The slope of the steepest segment also increases, with the number of pits to the 0.65 power.

Inside this region, the distribution appears chaotic and contains many non-linear structures. Figure 3 shows the stones for the pits 100 - 300 of the 8192-pit scenario. Notice the inverted parabolas and the cubic curve, highlighted in color. In a manner similar to that of the line segments, the parabolic segments become steeper closer to the home pit. When expressed as, $stones = a\,pit^2 + b\,pit + c$, the coefficient a begins at -1.5 near the threshold (pit 434 for this case) and decreases by increments of 1.5. Also, the number of pits involved in the parabolic segment generally decreases approaching the home pit. Note that many of the segments appear as parallel shells monotonically increasing or decreasing around a central segment that increases from near the axis and then returns to it.

In several cases, although not universally, higher-order polynomial segments were found. The first panel of Figure 3 shows a cubic segment found around pit 106. This suggests that simulations using more pits will reveal even higher degreed polynomial structures, but none were found in this work.

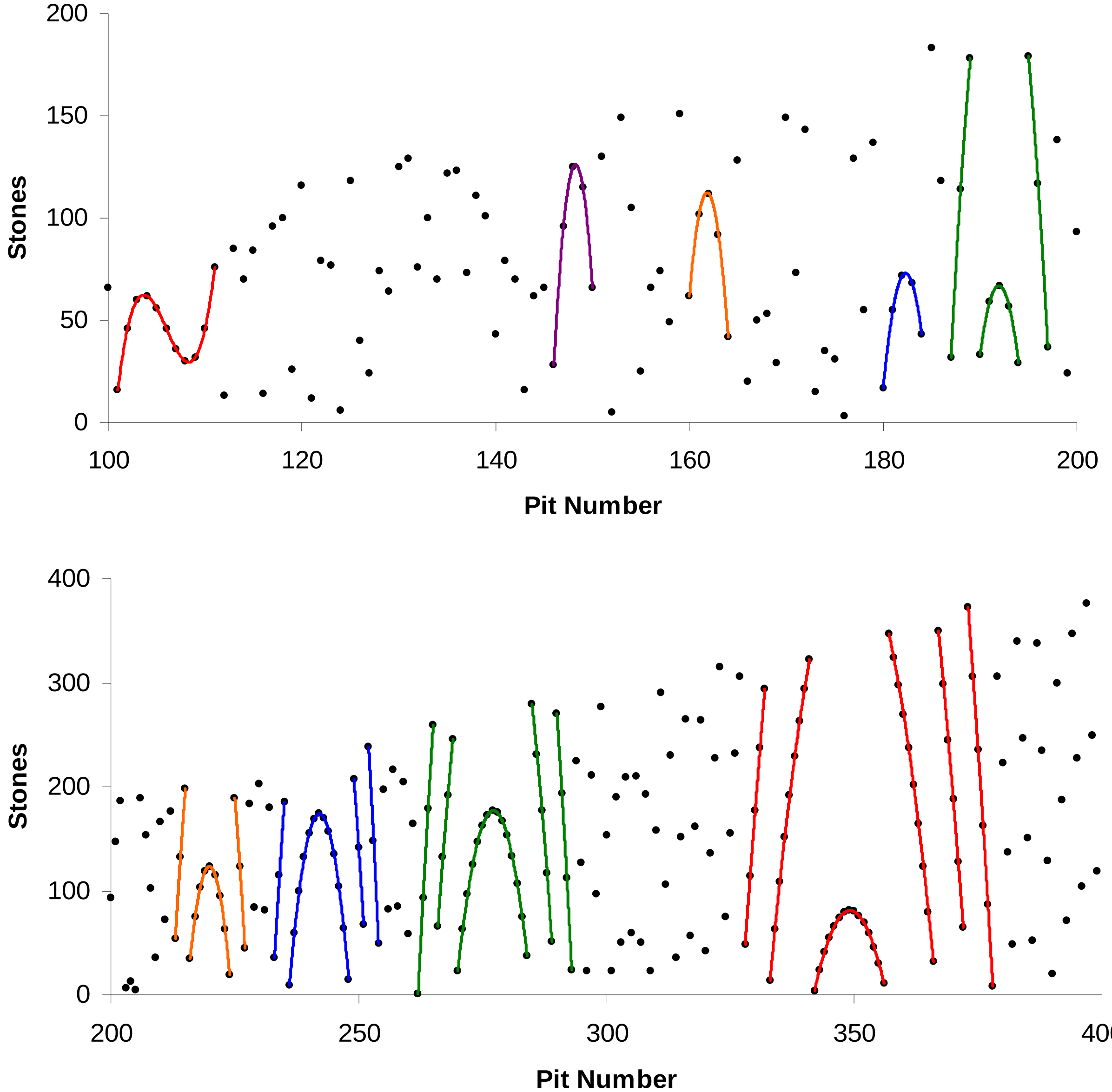

Figure 3: Non-Linear Portion of the 8192-Pit Distribution

Harvesting

The pit-harvesting sequence for the maximum number of stones in *n* filled pits is actually the reverse of OEIS entry A028920 [3]. For example, given six pits initially filled thusly (where pit 1 is the pit nearest the home pit):

Pit	6	5	4	3	2	1
Stones	6	4	2	3	1	1

the pits are harvested in this order: 1, 3, 1, 2, 1, 6, 1, 5, 1, 2, 1, 4, 1, 3, 1, 2, 1. This is the reverse of the first 17 terms of A028920 and is in keeping with the harvesting rule given by Broline [1].

To begin harvesting the maximum number of stones from a set of pits, each pit must contain between 1 and i stones, inclusive, where i is the pit number. To see this, consider a supposed maximal distribution where pit i is the lowest-numbered empty pit. Another winning distribution can be created by adding i stones to pit i and subtracting one stone each from pits 1 through $i-1$. This new distribution has one additional stone, so the original could not have been maximal. Also, if any pit i contains more than i stones, then harvesting it will overshoot the home pit, not enabling the next free move.

Based on an empirical study of harvesting maximal cases with up to 400 pits (51,449 moves), the frequency of occurrence of a specific value i in this sequence varies approximately with $1/i^2$. Specifically, the relative frequency f appears to be exactly

$$f = \frac{1}{i^2 + i}.$$

This implies that the ratio of the relative frequency of pit i to that of the previous pit is

$$\frac{f_i}{f_{i-1}} = \frac{i-1}{i+1}$$

Values are shown below for the first nine pits, which comprise 90% of the terms in the sequence.

Pit, *i*	**Relative Frequency, *f***	**1/*f***
1	0.5	2
2	0.166666667	6
3	0.083333333	12
4	0.05	20
5	0.033333333	30
6	0.023809524	42
7	0.017857143	56
8	0.013888889	72
9	0.011111111	90

Table 3: Relative Frequencies of Harvesting for the First Nine Pits

For a specific case of harvesting the maximal number of stones (S) from n pits, the sequence of pits to harvest is the reverse of the first S terms of A028920. This finite sequence always contains an odd number of terms and begins and ends with 1. Every odd-indexed term is 1, or every second move is to harvest from the first pit. This can be understood by realizing that, for every move, exactly one of the following two cases holds. Either:

1. The first pit contains one stone and is the lowest-numbered full pit and must be harvested. Alternatively,
2. The first pit is empty. Harvesting any other pit will deposit one stone into pit 1, filling it so that it will be harvested in the next move.

Relative to the first three pits, the harvesting sequence is periodic with period 12. Every maximal sequence (for more than three pits) has pits 1, 2, and 3 non-empty and begins with step 1, step 3, or step 7 (starred entries) in Table 4. This sub-sequence continues through step 12, at which point it repeats from step 1. Consequently, pit 2 is harvested every sixth turn and pit 3 every 12th turn.

Step	Pit 1	Pit 2	Pit 3	Harvest Pit
1*	1	1	1	1
2	0	1	1	> 3
3*	1	2	2	1
4	0	2	2	2
5	1	0	2	1
6	0	0	2	> 3
7*	1	1	3	1
8	0	1	3	3
9	1	2	0	1
10	0	2	0	2
11	1	0	0	1
12	0	0	0	> 3

Table 4: Periodic Sequence of Steps in Harvesting the First Three Pits

It appears that every pit number occurs regularly, as suggested by their relative frequencies. Higher pits, however, are not visited at constant intervals. For example, pit 4 is visited after 18 turns, then 20, then 22, 18, 20, 22, etc., for an average interval of 20. For pit 5, the chain of intervals is 26, 34, 26, 34, etc. In general, the chain of intervals seems to grow with the pit number. For pits 6 through 13, the numbers of intervals in the periodic chain is: 10, 15, 35, 28, 252, 210, 2310, and 900.

The pit-harvesting sequence appears to be a fractal sequence, but does not follow the strict definition of Kimberling [4]. The table below shows the northwest corner of the sequence's associative matrix. While the rows are generally interspersed, there are occasions of more than one element in a row appearing between two consecutive elements in a later row. For example, in row 4, both 180 and 234 are between the 178 and 238 in row 5. In a true fractal sequence, there would be exactly one element in each row between two subsequent elements in a lower row. Also, neither lower-trimming nor upper-trimming the sequence recovers the original, so it is not a fractal sequence, under the standard definitions.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
1	1	2	4	6	10	12	18	22	30	34	42
2	3	8	16	24	36	54	72	90	114	142	168

3	5	14	28	46	70	94	120	162	202	252	298
4	7	20	40	66	96	138	180	234	288	358	420
5	9	26	52	84	130	178	238	312	382	468	552
6	11	32	64	106	156	222	294	378	478	582	702
7	13	38	76	126	190	262	354	450	562	694	828
8	15	44	88	144	216	300	408	528	654	792	958
9	17	50	100	166	250	348	462	594	750	900	1080
10	19	56	112	186	276	390	522	660	834	1014	1218
11	21	62	124	204	310	432	574	738	922	1128	1350

Table 5: Upper-Left Corner of the Associative Matrix for the Pit-Harvesting Sequence

Conclusions

The game of Mancala is a fascinating combination of simple rules and complex play, particularly when taken to the case of an infinite board. For any number of pits, a maximal distribution of stones can be found that can be cleared by a succession of turns, each leaving the last stone in the home pit. Both the distribution of stones for a given number of pits and the pattern of moves in harvesting those stones is of interest, particularly as the number of pits becomes large.

It appears that the distribution of stones reaches a limiting pattern composed primarily of shorter and steeper linear segments, moving in toward the home pit. At some point, this pattern disintegrates into a mélange of higher-order polynomial segments. While not observed, it is speculated that, given enough pits, polynomial structures of arbitrarily high order will be found.

For any number of pits, the reverse of the harvesting sequence is the same previously-known sequence. Empirical study of this sequence shows regular patterns of visitation for pits 1, 2, and 3. For later pits, the pattern becomes increasingly complex, but still ultimately periodic. For all pits, the relative frequency of visitation appears to be given by a simple quadratic expression.

References

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2. Sloane, N. J. A., ed., Online Encyclopedia of Integer Sequences. http://www.research.att.com/~njas/sequences/A007952

3. Sloane, N. J. A., ed., Online Encyclopedia of Integer Sequences. http://www.research.att.com/~njas/sequences/A028920

4. Weisstein, Eric W. "Fractal Sequence." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/FractalSequence.html