Fun with Chaotic Orbits in the Mandelbrot Set

Kerry Mitchell
Mosaic Arts Center
16253 Yavapai Street
Goodyear, AZ 85338 USA
E-mail: lkmitch@gmail.com

Abstract

The orbits of points in the Mandelbrot set can exhibit chaos, if the point is on the boundary and not part of a periodic disk. Chaotic orbits that begin nearby are known to diverge exponentially; this characteristic can be exploited to combine multiple chaotic orbits into one image. Such combinations can be used to investigate the dynamics of the orbit or enjoyed for their own aesthetic sake.

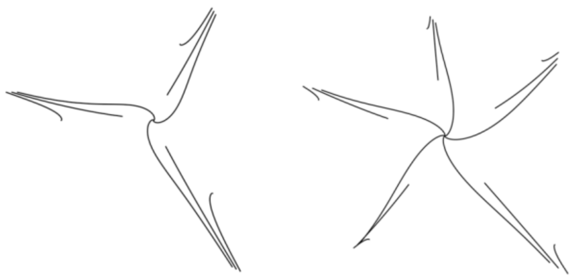
Main Cardioid Boundary Points

The iconic shape of the Mandelbrot set is of a main cardioid surrounded by an infinitude of disks, each tangent to the cardioid at a single point. To simplify further explanations, use the standard definition of the Mandelbrot set as the set of complex points *c*, such that the iteration $z = z^2 + c$ is bounded for all iterations, beginning with $z = 0$. The orbit of *c* is simply the sequence of iterates, *z*. Points inside the set's main cardioid have orbits that converge to a final fixed point (e.g., $c = 0$, whose orbit is $z = 0$ for all iterations). Points inside a tangent disk have orbits that converge to a cycle of values (e.g., $c = -1$, whose orbit oscillates between $z = -1$ and $z = 0$). Right on the boundary, neither effect dominates. If the point is the tangent point of a disk, then the orbit will slowly settle to a fixed point, but in an oscillatory fashion (e.g., the tangent point of the period-two disk, $c = -0.75$, whose orbits settles to $z = -0.5$, but through two distinct branches). For a deeper examination of the boundary of the Mandelbrot set, see, for example, Peitgen [1].

In [1], boundary points *c* are characterized by their angle around the origin, θ:

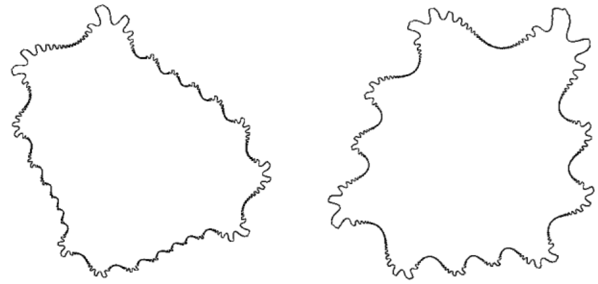
$$c = \left(\frac{1-\cos(\theta)}{2}\right)[\cos(\theta) + i\ sin(\theta)] + \frac{1}{4}.$$

If θ is a rational multiple (*m*/*n*, for *m* and *n* both positive integers and in lowest terms) of 2π radians, then *c* will be the point of tangency of an *n*-period disk. That is, the orbit of *c* will settle into a cycle of *n* values. This is illustrated in the left panel of Figure 1, showing the orbits for points near the tangent point of the 1/3 disk (located at the top of the main cardioid). There are three orbits shown, each having three branches, indicating the periodicity of the disk. In each branch, the clockwise-most curves are the orbit for a point just inside the cardioid's edge. The orbit converges to a fixed point, near the center of the image. The counter-clockwise-most curves are for a point just outside of the cardioid, inside the disk. This orbit converges to a limit cycle of three values, so the curves stop at three points. In between these two sets is the boundary point's orbit, which slowly moves toward a fixed point, but still retains the period-3 nature. The right panel shows the corresponding orbits for points near the tangent point of the 2/5 disk.



1/3 disk *2/5 disk*

Figure 1: *Orbits near points of disk tangency.*

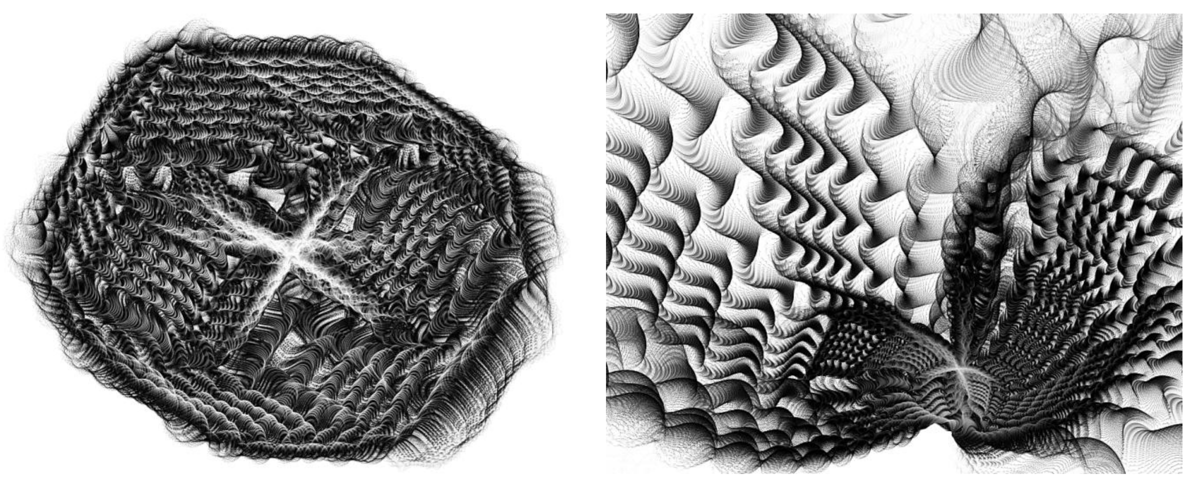


θ = 2 radians *θ = 4 radians*

Figure 2: *Chaotic orbits of boundary points that are not tangent points.*

If the angle of *c* is not a rational multiple of 2π radians, then *c* is not the point of tangency of a disk. Indeed, *c*'s orbit is chaotic, bounded but neither periodic nor converging to a fixed point. Such points are easily found; since π is an irrational number, any rational multiple of π will be an irrational number. Thus, using a rational value for θ (in radians) will lead to a boundary point *c* that is not the tangent point of any disk. The orbit of *c* will have a fractal structure, as illustrated in Figure 2 for θ = 2 and 4 radians. Despite their appearance, these orbit paths are not simple closed curves and the orbits are not periodic.

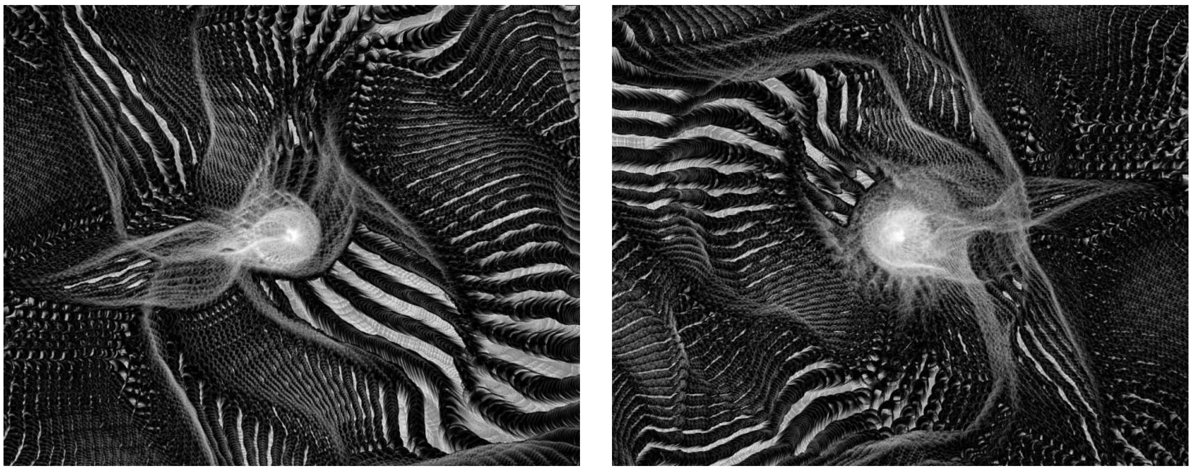
It is well known that if a dynamical system exhibits chaos, then points that are initially very close will diverge exponentially (Moon [2]). This characteristic, combined with the bounded nature of boundary point orbits, results in a novel method of creating images. Take two different non-tangent boundary points, compute their orbits, and plot some combination of them (e.g., their difference or their ratio). Since the orbits diverge but are constrained, their combination might be expected to fill a region of the plane in interesting ways. This is indeed the case; in Figure 3, the difference and the ratio of the orbits for boundary points at θ = 1.999 and 2.001 radians are shown. These and further images were created using the program Ultra Fractal [3], employing a coloring routine written by the author. Grayscale levels were computed by logarithmically scaling the number of times the combination of orbits (e.g., difference or ratio) landed on the given pixel.



Difference *Ratio*

Figure 3: *Combining two chaotic orbits.*

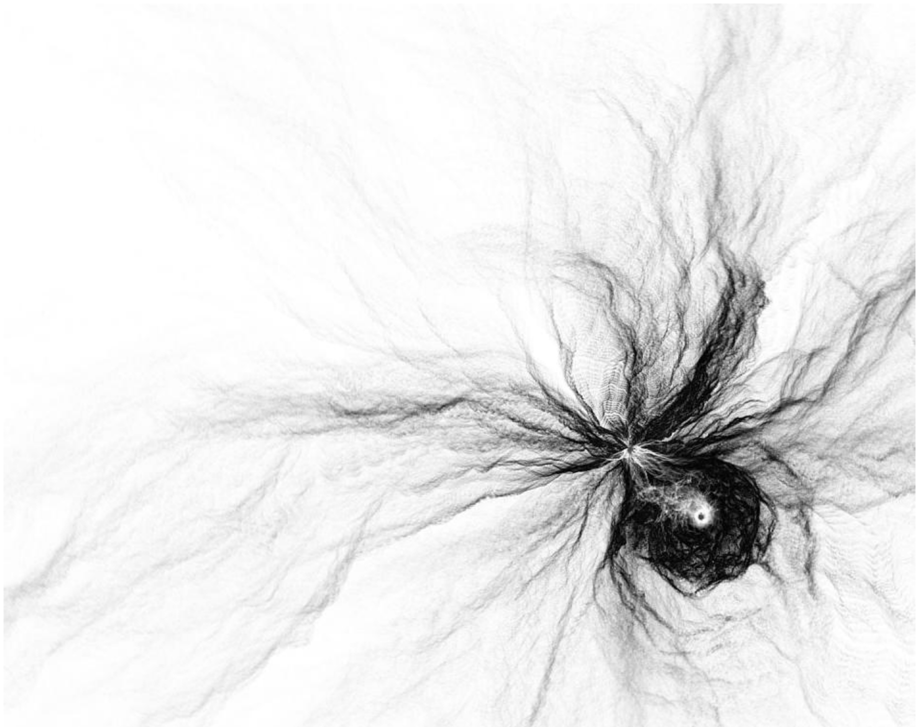
Each point's orbit can be considered a discrete sample. Taking appropriate combinations of multiple samples can then approximate a derivative of the orbit with respect to polar angle. For example, the finite difference shown in Figure 3 is a scaled version of the first derivative of the orbit at $\theta = 2$ radians, using a step size of 0.002 radians. Other derivatives can be easily approximated, using standard finite difference equations (see, for example, Gerald [4]). Figure 4 shows approximations to the second and fourth derivatives of the orbit at $\theta = 5$ radians.



Second derivative *Fourth derivative*

Figure 4: *Approximating orbit derivatives*

"Simple as the Number Nine"



"Black Widower"

Figure 5: *Artistic uses of the cardioid boundary orbits coloring method.*

Finally, this method can be expanded to create intriguing images, for their own sake. Modifications include: modifying the values before combining them, varying the weights of individual orbits, and using non-standard means of combining orbits. The author employed such techniques, among others, in creating the works, "Simple as the Number Nine" [5] and "Black Widower," [6] shown in Figure 5.

References

[1] H-O. Peitgen, H. Jürgens, and D. Saupe, *Chaos and Fractals: New Frontiers of Science*, Springer-Verlag, pp. 855–875. 1992.

[2] F.C. Moon, *Chaotic Vibrations: An Introduction for Applied Scientists and Engineers*, John Wiley & Sons, pp. 191–195. 1987.

[3] F. Slijkerman, *Ultra Fractal: Advanced Fractal Animation Software*, www.ultrafractal.com.

[4] C.F. Gerald, P.O. Wheatley, *Applied Numerical Analysis*, sixth edition, Addison-Wesley, pp. 373–374. 1999.

[5] L.K. Mitchell, *Simple as the Number Nine,* www.kerrymitchellart.com/gallery23/simple.html.

[6] L.K. Mitchell, *Black Widower*, www.kerrymitchellart.com/gallery23/blackwidower.html.